6.3 Ecological Models

predator-prey system (Lotka-Volterra system)

$x(t)$: prey (rabbits)

$y(t)$: predator (wolves) predator uses prey as food source

$$\frac{dx}{dt} = ax - pxy \qquad a, b, p, q > 0$$

$$\frac{dy}{dt} = -by + qxy$$

notice if there no predator, $\frac{dx}{dt} = ax$ exponential growth

if there no prey, $\frac{dy}{dt} = -by$ exponential decay

$\frac{dx}{dt} = x(a - py)$ act as reduction to growth rate

$\frac{dy}{dt} = y(-b + qx)$ act as boost to growth rate cp: $(0,0)$, $\left(\frac{b}{g}, \frac{a}{p}\right)$

Jacobian: $J(x,y) = \begin{bmatrix} a-py & -px \\ gy & -b+gx \end{bmatrix}$

Goal: given $x(0)$, $y(0)$, what happens to $x(t)$, $y(t)$ as $t \to \infty$?

<u>example</u>

$$\frac{dx}{dt} = x - 0.5xy = x(1-0.5y)$$

$$\frac{dy}{dt} = -0.75y + 0.25xy = y(-0.75 + 0.25x)$$

cp: $\underbrace{(0,0)}_{\text{everybody dies}}$, $\underbrace{(3,2)}_{\text{stable pop.}}$

$$J(x,y) = \begin{bmatrix} 1-0.5y & -0.5x \\ 0.25y & -0.75+0.25x \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -0.75 \end{bmatrix}$$

$\lambda = 1, -0.75$

saddle
unstable
not sensitive to perturbation

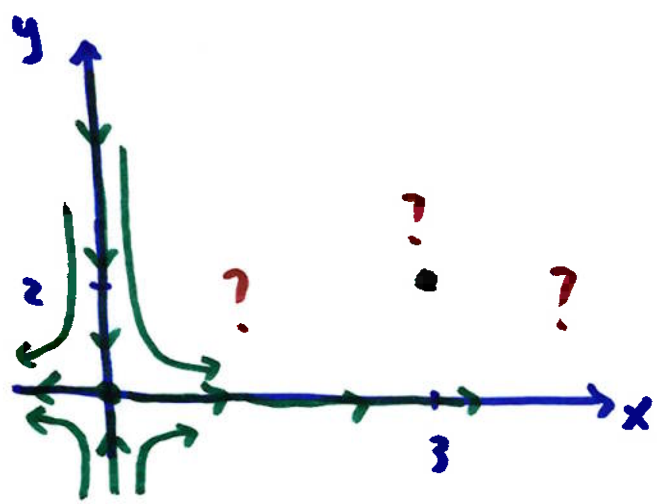
$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Solutions near (0,0) $\vec{x}(t) = c_1 e^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-0.75t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

if initial condition leads to $c_1 = 0$,

then $\vec{x}(t) = c_2 e^{-0.75t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ into (0,0) along y-axis

likewise, we see solutions go away from (0,0) along x-axis

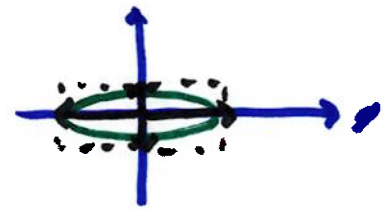


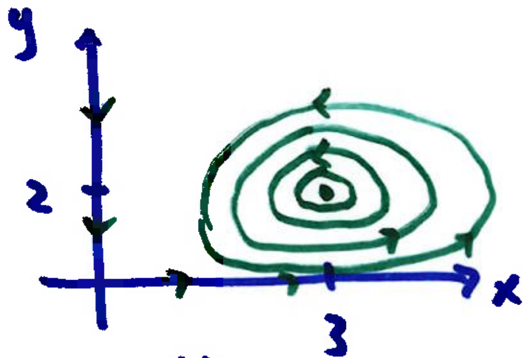
as long as $x(0) \neq 0$
then solutions do NOT go into (0,0)

$$J(3,2) = \begin{bmatrix} 0 & -1.5 \\ 0.5 & 0 \end{bmatrix} \qquad \lambda = \pm \frac{\sqrt{3}}{2} i$$

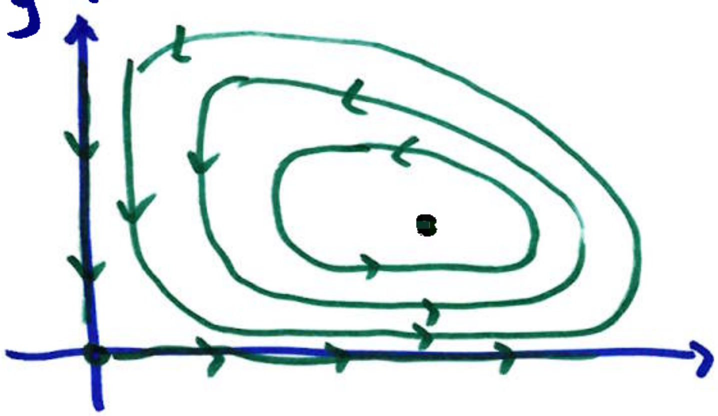
center
stable
sensitive to perturbation
(subject verification)

$$\vec{v} = \begin{bmatrix} 1 \\ \pm \frac{1}{\sqrt{3}} i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pm i \begin{bmatrix} 0 \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$



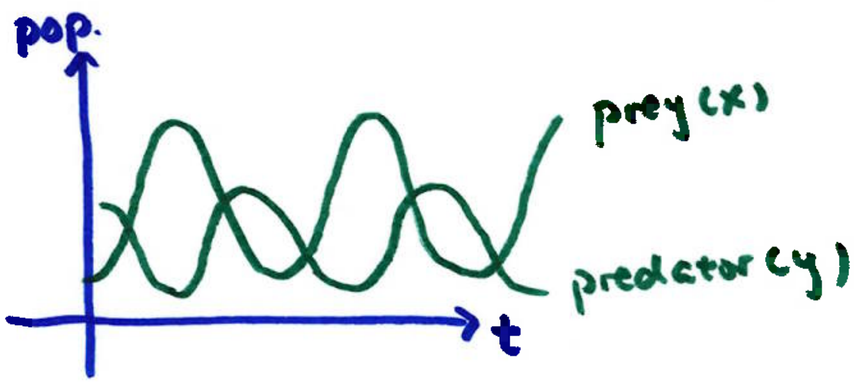


can use the flow from saddle pt to find direction of ellipse



in nonlinear picture the ellipses are generally warped.

x increases, then more food for y so it increases a bit later, then lots of y eating x so x declines, which means less food for y so y declines, so x has chance to increase and then the cycle repeats.



$$x' = ax - pxy$$
$$y' = -by + \gamma xy$$

period of variation is $\frac{2\pi}{\sqrt{ab}}$ independent of $x(0), y(0)$

y lags behind x by a $\frac{1}{4}$ period

amplitude of x is $\frac{kb}{\gamma}$

k : depends on $x(0), y(0)$

" " y is $\frac{k\sqrt{ab}}{p}$

verifying (3, 2) is a center: computer graph

or solve

$$\left.\begin{aligned} x' &= ax - pxy \\ y' &= -by + gxy \end{aligned}\right\} \quad \frac{dy}{dx} = \frac{y(-b+gx)}{x(a-py)}$$

$$\vdots$$

$$a \ln y - py + b \ln x - gx = C$$

warped ellipses.